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An Example in Tangential Meromorphic Approximation

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It is known that all sets of meromorphic uniform approximation in $\mathbf{C}$ which satisfy an additional condition involving the Gleason parts of the algebra $R(K)$ are then also sets of tangential approximation by meromorphic functions. In this paper, we construct a set which, although it is a set of tangential approximation, does not satisfy this extra condition on parts and, thus, showing that the condition fails to be necessary. Finding a complete characterization of sets of meromorphic tangential approximation is still an open problem.

1. INTRODUCTION

Let Ω be a domain in the extended complex plane $\overline{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$, with $\Omega \neq \overline{\mathbf{C}}$. A relatively closed subset $E \subset \Omega$ is called a *set of holomorphic* (respectively *meromorphic*) *tangential* (or *Carleman*) *approximation* if, for any pair of functions f and ε, with f continuous on E and holomorphic in its interior E^0, and with ε continuous and positive on E, there exists a function g holomorphic (respectively meromorphic) on Ω such that

$$|f - g| < \varepsilon \quad \text{on } E. \tag{1}$$

In particular, ε can tend to zero arbitrarily rapidly as we approach the boundary of Ω along E. In 1927, Carleman [2] showed that the real axis is a Carleman set of approximation by holomorphic functions in $\mathbf{C}$, and

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Nersessian [8] gave in 1971 a complete characterization of sets of holomorphic Carleman approximation in the plane. This result was later extended to arbitrary non-compact Riemann surfaces by Boivin [1], where meromorphic Carleman approximation was also studied.

Let us now describe some of these results. In addition to necessarily being sets of uniform approximation (that is when ε is a constant function in (1)), it was shown by Gauthier [6] that sets of Carleman approximation must also satisfy the following condition $\mathcal{G}$ (that these two conditions are also sufficient is Nersessian's theorem [8]):

DEFINITION. Let E be a relatively closed subset of Ω. If for every compact set $X \subset \Omega$, there is a compact set Y, $X \subset Y \subset \Omega$ such that no component of the interior of E (respectively no component of the *fine* interior of E, or respectively no (Gleason) *part* of E) meets both X and $\Omega \setminus Y$, we then say that E satisfies condition $\mathcal{G}$ (respectively $\mathcal{G}_F$ or $\mathcal{G}_P$).

The fine topology is the coarsest topology on Ω for which all superharmonic functions on Ω are continuous. To define parts, let $C(E)$ denotes the complex-valued continuous functions on E with the usual supremum norm

$$\|f\|_E = \sup_{z \in E} |f(z)|, \qquad f \in C(E),$$

and let $M(E)$ denote the uniform closure in $C(E)$ of the functions meromorphic in Ω without poles on E. When E is compact, the notation $R(E)$ is also commonly used. We define the *parts* of E to be the Gleason parts relative to the algebra $R(E)$, when E is compact (for the definition of Gleason parts relative to a uniform algebra, see [5, Chap. VI]). When E is closed, but not necessarily compact, the parts of E are defined by means of an exhaustion of Ω by compact subsets (see [1, Definitions 5 and 6]). Note that $\mathcal{G}_P \Rightarrow \mathcal{G}_F \Rightarrow \mathcal{G}$. It was shown in [1] that $\mathcal{G}_F$ is also a necessary condition for meromorphic (or holomorphic) Carleman approximation, and that uniform meromorphic approximation plus condition $\mathcal{G}_P$ implies meromorphic Carleman approximation. The complete characterization of sets of meromorphic Carleman approximation is still an open problem.

In this paper, we answer a question raised in [1] by showing that there exists a set with an unbounded part which is nonetheless a set of meromorphic Carleman approximation. In other words, it will be shown that condition $\mathcal{G}_P$, though sufficient, is not necessary.

We end this section with some more notations. $\bar{S}$, ∂S, and S^0 will denote the closure, boundary and the interior respectively of a subset S of the complex plane $\mathbf{C}$. $A(X)$ will be the class of continuous functions on the closed subset $X \subset \mathbf{C}$ which are holomorphic on X^0. Finally we let $\Delta = \Delta(0, 1) = \{z \in \mathbf{C}\colon |z| < 1\}$ and more generally $\Delta(a, r) = \{z \in \mathbf{C}\colon |z - a| < r;\ r > 0\}$.

2. PRELIMINARIES

Following [7] we introduce two definitions.

DEFINITION 2. We will call a closed domain of the form

$$P = P(\{z_i\}; \{r_i\}) = \bar{\Delta} \setminus \bigcup_{i=1}^{\infty} \Delta(z_i, r_i)$$

an *L-set*, if the sequences $\{z_i\}$ and $\{r_i\}$ satisfy the conditions:

(a) The set of cluster points of the sequence $\{z_i\}$ is equal to $\partial \Delta$;
(b) $r_i < 1 - |z_i|$;
(c) $r_i + r_j < |z_i - z_j|$, when $i \neq j$.

DEFINITION 3. We will call an L-set P a *uniqueness set*, if any function $f \in A(P)$, which is equal to zero on $\partial \Delta$, is equal to zero identically. If there is a function $f \in A(P)$ such that $f(z) = 0$ for $z \in \partial \Delta$ and $f(z) \not\equiv 0$, $z \in P$, then P is said to have the property of *nonuniqueness*.

In [7], Gonchar has shown the existence of an L-set P_0 which is a set of nonuniqueness and which has the following additional property:

$$\sum_{i=1}^{\infty} r_i^{\alpha} < \infty \qquad \text{for any} \quad \alpha > 0. \tag{2}$$

It is also important for us that in [7], the function μ constructed to demonstrate the nonuniqueness property of the L-set P_0 satisfying (2) is in fact a meromorphic function in Δ having poles only at the points of the sequence $\{z_i\}$. Let

$$C_1 = \{z\colon |z| = 1, \ -1 \leqslant \mathcal{R}z < 0\}, \qquad C_2 = \{z\colon |z| = 1, 0 < \mathcal{R}z \leqslant 1\}.$$

We first prove:

LEMMA 1. *For the nonuniqueness set P_0 above, there exists a function v meromorphic in the unit disc such that*

$$\lim_{P_0 \ni z \to \zeta} v(z) = \begin{cases} 0, & \text{if} \quad \zeta \in C_1, \\ 1, & \text{if} \quad \zeta \in C_2. \end{cases} \tag{3}$$

Proof. Denote

$$A_1 = P_0 \setminus (\Delta(1; \sqrt{2}) \cup \partial \Delta),$$
$$A_2 = P_0 \setminus (\Delta(-1; \sqrt{2}) \cup \partial \Delta),$$
$$F = A_1 \cup A_2.$$

Consider the function

$$f=\begin{cases}0 & \text{on } A_1\\ 1 & \text{on } A_2.\end{cases}$$

For an arbitrary closed disc $D\subset\Delta$ the complement of the intersection $F\cap D$ consists of only finitely many components (since the cluster set of $\{z_i\}$ is $\partial\Delta$), hence F is a set of uniform approximation by meromorphic functions in Δ according to a theorem of Nersessian ([9, 10]; see also [4]).

Consider the function f/μ. According to Mittag–Leffler's theorem there is a function h, meromorphic in Δ having poles only at zeros of the function μ in A_2 with principal parts coinciding with those of $1/\mu$. Thus we can assume that $(f/\mu)-h\in A(F)$. According to the theorem of Nersessian there exists a function g meromorphic in Δ such that

$$\left|\left(\frac{f}{\mu}-h\right)-g\right|<1 \qquad \text{on } F.$$

It follows that we have $|f-\mu(h+g)|<|\mu|$ on F. Since $\mu\to 0$ when $z\to\zeta\in\partial\Delta$, $z\in P_0$, the function $v=\mu(h+g)$ is the needed one. This proves the lemma.

3. CONSTRUCTION OF THE EXAMPLE

The main result of the present note is the following.

THEOREM 1. *There exists a closed subset E of the complex plane* **C**, *such that E^0 belongs to a single unbounded Gleason part of $M(E)$ and such that E is a set of tangential approximation by meromorphic functions.*

Proof. Consider the strip

$$\Pi=\{z=x+iy:\ -1\leqslant y\leqslant 1\}.$$

We take the nonuniqueness L-set

$$P_0=\overline{\Delta(0,1)}\setminus\bigcup_{i=1}^{\infty}\Delta(z_i,r_i)$$

satisfying condition (2) found by Gonchar and we denote

$$E=\left[\Pi\Big\backslash\bigcup_{n=-\infty}^{\infty}\bigcup_{i=1}^{\infty}\Delta(z_i+3n,r_i)\right]\Big\backslash\bigcup_{n=-\infty}^{\infty}\Delta(3n\pm i,\tfrac{1}{4});$$

$$D'_n=\overline{\Delta(0,3n)}\setminus(\Delta(\pm 3n,1)\cup\Delta(\pm 3n\pm i,\tfrac{1}{4})),\qquad n\geqslant 1;$$

$$D''_0=\varnothing;\qquad D''_n=(\overline{\Delta(0,3n)}\cup\overline{\Delta(\pm 3n,1)})\setminus\Delta(\pm 3n\pm i,\tfrac{1}{4}),\qquad n\geqslant 1.$$

Notice, that in the following discussions condition (2) is applied only with $\alpha = 1$. The arguments presented in [3; 5, Chap. VIII, Section 10] to produce a counterexample in the theory of bounded pointwise approximation along with condition (2) show that the interior of $E \cap \overline{\Delta(0, 3/2)}$ is not a set of bounded pointwise appproximation by rational functions. We claim that this implies that $U_1 = (P_0 \cap E)^0$ and $U_2 = [(E^0 \cap \Delta(0, 3/2)] \backslash P_0$ belong to the same Gleason part of $R(X)$, where $X = E \cap \overline{\Delta(0, 2)}$.

To prove the claim, we first notice that both U_1 and U_2 are sets of bounded pointwise approximation by rational functions. This follows for example from [5, Corollary VIII.11.2; 5, Theorem VI.5.3], respectively. This means that if f is an arbitrary bounded and holomorphic function on $U_1 \cup U_2$, there exist sequences $\{r_n^{(1)}\}$ and $\{r_n^{(2)}\}$ of rational functions without poles on $\overline{U_1}$ and on $\overline{U_2}$, respectively, such that $\|r_n^{(i)}\|_{\overline{U_i}} \leqslant \|f\|_{U_i}$, $i = 1, 2$, and such that $r_n^{(i)}$ converges pointwise to f on U_i. Note that the functions $r_n^{(i)}$ can be chosen without poles on X, and thus can be taken to be bounded on X.

Assume now that U_1 and U_2 are contained in two different Gleason parts of $R(X)$. Using Lemma 2 in [1, Section 5] to expand slightly the set U_2, it then follows from the Lemma 0–1 in [1, Section 2] that there exist a sequence $\{\varphi_n\}$, $\varphi_n \in R(X)$, $\|\varphi_n\|_X \leqslant 1$ such that, for $n > 2$,

$$|\varphi_n| \leqslant \frac{1}{n\,\|r_n^{(2)}\|_X} \qquad \text{on} \quad U_2 \backslash \Delta\left(0, 1 + \frac{1}{n}\right)$$

and

$$|1 - \varphi_n| \leqslant \frac{1}{n\,\|r_n^{(1)}\|_X} \qquad \text{on} \quad U_1 \cap \Delta\left(0, 1 - \frac{1}{n}\right).$$

It follows that $q_n := \varphi_n r_n^{(1)} + (1 - \varphi_n)\, r_n^{(2)} \in R(X)$ and that q_n converges pointwise to f on $U_1 \cup U_2$, with $\|q_n\|_{\overline{U_1 \cup U_2}} \leqslant C \|f\|_{U_1 \cup U_2}$ and so $U_1 \cup U_2$ is a set of bounded pointwise approximation by rational functions, a contradiction. Therefore U_1 and U_2 must belong to the same Gleason part of $R(X)$. From this, and from the definition of the set E, it can now be easily deduced that the intersection of E^0 with any compact disc D is contained in a single Gleason part for the corresponding Banach algebra $R(E \cap D)$.

Note that this last assertion follows also from B. Øksendal's criterion of peak sets and Gleason parts for the algebra $R(X)$ for certain sets with smooth boundaries [11]. The capacity condition at all boundary points of P on the unit circle in that criterion follows from the condition (2) in the following way. If $x \in \partial\Delta$, thanks to known estimates for the analytic capacity and the eventual uniform distribution of the deleted discs in the construction of the set P_0 (see for details [7, 11, 5]), we will have

$$\lim_{r\to 0^+}\frac{\gamma(\Delta\cap\Delta(x,r)\backslash P_0)}{r}\leqslant\lim_{r\to 0^+}\frac{6\pi r\sum_{|z_i|>1-r}r_i}{2\pi r}$$

$$=3\lim_{r\to 0^+}\sum_{|z_i|>1-r}r_i=0.$$

We have thus established that E^0 belongs to a single Gleason part of $M(E)$. Further, we remark that since all inner boundary points of the set E belong to circular arcs, according to known criteria for uniform rational approximation due to Vitushkin (see [5, Chap. VIII, Sect. 13]), any compact portion of the set E is a set of uniform rational approximation.

To show that E is also a set of tangential approximation by meromorphic functions, we will need the following result (compare with the lemma in [8] and with Lemma 0–1 in [1]).

LEMMA 2. *For each $n\geqslant 1$ and arbitrary positive number δ there exist rational functions ρ_n such that*

$$\begin{aligned}&\text{(i)}\quad |\rho_n|<\delta &&\text{on}\quad D'_n,\\&\text{(ii)}\quad |\rho_n-1|<\delta &&\text{on}\quad (D''_{n+1}\backslash(D''_n)^0)\cap E,\\&\text{(iii)}\quad |\rho_n|<c &&\text{on}\quad (D''_{n+1}\cap E)\cup D'_n,\end{aligned}\tag{4}$$

for $c=\|v\|_{P_0}+2$, where v is the function constructed in Lemma 1.

Proof of the lemma. We will use the function h defined as follows:

$$h=\begin{cases}0 & \text{on}\quad D'_n,\\ 1-v & \text{on}\quad \overline{\Delta(-3n,1)}\backslash\Delta(-3n\pm i,\tfrac{1}{4}),\\ v & \text{on}\quad \overline{\Delta(3n,1)}\backslash\Delta(3n\pm i,\tfrac{1}{4}),\\ 1 & \text{on}\quad (D''_{n+1}\backslash D''_n)\cap E.\end{cases}\tag{5}$$

According to the above-mentioned criterion of Vitushkin, the set $F_n=D''_n\cup((D''_{n+1}\backslash D''_n)\cap E)$ is a set of uniform rational approximation. Applying uniform rational approximation to the function h on F_n within appropriate bounds we obtain the desired rational function ρ_n. This completes the proof of the lemma.

We can now proceed with the proof of the approximation part of the theorem. Let $f\in A(E)$ be an arbitrary function and ε be an arbitrary positive function continuous on E tending to zero when its argument approaches ∞.

Step I. There is a rational function R_1 such that

$$|f - R_1| < \frac{\varepsilon(4)}{4c} \qquad \text{on} \quad D''_1 \cap E. \tag{6}$$

Step II. Choose a rational function Q_2 without poles on D'_1 and such that

$$|f - R_1 - Q_2| < \frac{\varepsilon(7)}{4c} \qquad \text{on} \quad (D''_2 \backslash (D'_1)^0) \cap E. \tag{7}$$

Assume now that δ in Lemma 2 is chosen so small that the function $R_2 = \rho_1 Q_2$ satisfies the conditions:

$$|R_2| < \frac{1}{2^2} \qquad \text{on} \quad D'_1,$$

$$|f - R_1 - R_2| < \varepsilon(4) \qquad \text{on} \quad D'_1 \cap E,$$

$$|f - R_1 - R_2| < \frac{\varepsilon(7)}{4c} \qquad \text{on} \quad (D''_2 \backslash D''_1) \cap E.$$

Besides, we have on the set $(D''_1 \backslash D'_1) \cap E$, thanks to Eqs. (6), (7), and (4iii):

$$\begin{aligned} |f - R_1 - R_2| &\leqslant |f - R_1| + |R_2| \\ &< \frac{\varepsilon(4)}{4c} + c\,|Q_2| \\ &< \frac{\varepsilon(4)}{4c} + c\left(\frac{\varepsilon(4)}{4c} + \frac{\varepsilon(7)}{4c}\right) \\ &< \frac{\varepsilon(4)}{4} + \frac{\varepsilon(4)}{2} < \varepsilon(4). \end{aligned}$$

Step III. Assume now that for some $n \geqslant 2$ the functions $R_1, ..., R_n$ are constructed in such a way that

$$\text{(i)} \quad |R_k| < \frac{1}{2^k} \qquad \text{on} \quad D'_{k-1}, \quad \text{for} \quad k = 2, ..., n,$$

$$\begin{aligned} \text{(ii)} \quad & |f - R_1 - \cdots - R_n| < \varepsilon(3k+1) \\ & \qquad \text{on} \quad (D''_k \backslash D''_{k-1}) \cap E, \quad \text{for} \quad k = 1, ..., n-1, \end{aligned} \tag{8}$$

$$\text{(iii)} \quad |f - R_1 - \cdots - R_n| < \frac{\varepsilon(3n+1)}{4c} \qquad \text{on} \quad (D''_n \backslash D''_{n-1}) \cap E.$$

There is a rational function Q_{n+1} without poles on D'_n such that

$$|f - R_1 - \cdots - R_n - Q_{n+1}| < \frac{\varepsilon(3(n+1)+1)}{4c} \qquad \text{on} \quad (D''_{n+1} \backslash (D'_n)^0) \cap E. \tag{9}$$

Taking δ in Lemma 2 sufficiently small we can assure that the rational function $R_{n+1} = \rho_n Q_{n+1}$ satisfies the conditions:

$$\text{(i)} \quad |R_{n+1}| < \frac{1}{2^{n+1}} \qquad \text{on} \quad D'_n,$$

$$\text{(ii)} \quad |f - R_1 - \cdots - R_{n+1}| < \varepsilon(3k+1)$$

$$\text{on} \quad (D''_k \backslash D''_{k-1}) \cap E, \quad \text{for} \quad k = 1, ..., n-1, \tag{10}$$

$$|f - R_1 - \cdots - R_{n+1}| < \frac{\varepsilon(3n+1)}{4c} \qquad \text{on} \quad (D'_n \backslash D''_{n-1}) \cap E,$$

$$\text{(iii)} \quad |f - R_1 - \cdots - R_{n+1}| < \frac{\varepsilon(3(n+1)+1)}{4c} \qquad \text{on} \quad (D''_{n+1} \backslash (D''_n)^0 \cap E.$$

For the points $z \in (D''_n \backslash D'_n) \cap E$, according to Eqs. (8iii), (9), and (4iii) we have

$$\begin{aligned} |f - R_1 - \cdots - R_{n+1}| &\leqslant |f - R_1 \cdots - R_n| + |R_{n+1}| \\ &< \frac{\varepsilon(3n+1)}{4c} + c\left(\frac{\varepsilon(3(n+1)+1)}{4c} + \frac{\varepsilon(3n+1)}{4c}\right) \\ &< \varepsilon(3n+1). \end{aligned} \tag{11}$$

Step IV. It follows from Eqs. (10) and (11) that Eqs. (8), with $n+1$ in place of n, are satisfied for the functions $R_1, ..., R_{n+1}$. We have thus shown by induction that Eqs. (8) are satisfied for a sequence $\{R_n\}_{n=1}^{\infty}$ of rational functions; that is, the following estimations are valid:

$$\text{(i)} \quad |R_k| < \frac{1}{2^k} \qquad \text{on} \quad D'_{k-1} \quad \text{for} \quad k = 2, 3, ...,$$

$$\text{(ii)} \quad |f - R_1 - \cdots - R_n| < \varepsilon(3k+1) \tag{12}$$

$$\text{on} \quad (D''_k \backslash D''_{k-1}) \cap E, \quad \text{for} \quad k = 1, ..., n; \quad n = 1, 2, ...$$

Thanks to (12i) the series

$$G = \sum_{n=1}^{\infty} R_n$$

represents a function meromorphic in **C** which satisfies the approximation estimate

$$|f-G| \leqslant \varepsilon \qquad \text{on} \quad E.$$

The last assertion follows from (12ii) by first fixing an arbitrary point $z \in E$, choosing and fixing the unique $k_0 \geqslant 1$ with $z \in (D''_{k_0} \backslash D''_{k_0-1}) \cap E$, and then passing to the limit as $n \to \infty$. This completes the proof of the theorem.

REFERENCES

1. A. Boivin, Carleman approximation on Riemann surfaces, *Math. Ann.* **275** (1986) 57–70.
2. T. Carleman, Sur un théorème de Weierstrass, *Ark. Mat. Astron. Fys. B* **20**, No. 4 (1927), 1–5.
3. S. Fisher, Bounded approximation by rational functions, *Pacific J. Math.* **28** (1969), 319–326.
4. D. Gaier, "Vorlesungen über Approximationen im Komplexen," Birkhäuser, Basel, (1980); English transl., "Lectures on Complex Approximation," Birkhäuser, Basel, 1987.
5. T. Gamelin, "Uniform Algebras," 2nd ed., Chelsea, New York, 1984.
6. P. M. Gauthier, Tangential approximation by entire functions and functions holomorphic in a disc, *Izv. Akad. Nauk Armenii Mat.* **4** (1969), 319–326.
7. A. A. Gonchar, On examples of non-uniqueness of analytic functions, *Vestnik Moskov. Univ. Ser. I Mat. Mech.* **1964**, No. 1 (1964), 37–43. [in Russian]
8. A. H. Nersessian, On Carleman sets, *Izv. Akad. Nauk Armenii Mat.* **6** (1971), 465–471 [in Russian]; English transl., *AMS Transl. (2)* **122** (1984), 99–104.
9. A. H. Nersessian, On uniform and tangential approximation by meromorphic functions, *Izv. Akad. Nauk Armenii Mat.* **7** (1972), 406–412 [in Russian]; English transl., *AMS Transl. (2)* **144** (1989), 71–77.
10. A. H. Nersessian, "Certain Problems in the Theory of Approximation," dissertation, Yerevan State University 1974. [in Russian]
11. B. Øksendal, Null sets for measures orthogonal to $R(X)$, *Amer. J. Math.* **94** (1972), 331–342.